Infrared Zero of β and Value of γ_m for an SU(3) Gauge Theory at the Five-Loop Level

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We calculate the value of the coupling at the infrared zero of the beta function of an asymptotically free SU(3) gauge theory at the five-loop level as a function of the number of fermions. Both a direct analysis of the beta function and analyses of Padé approximants are used for this purpose. We then calculate the value of the five-loop anomalous dimension, γ_m, of the fermion bilinear at this IR zero of the beta function.

The evolution of an asymptotically free gauge theory from the ultraviolet (UV) to the infrared (IR) is of fundamental importance. The evolution of the running gauge coupling $g = g(\mu)$, as a function of the Euclidean momentum scale, μ, is described by the renormalization-group (RG) beta function [1], $\beta_g = dg/dt$ or equivalently, $\beta = d\alpha/dt = [g/(2\pi)]\,\beta_g$, where $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $dt = d\ln\mu$ (the argument μ will often be suppressed in the notation). Here we consider a vectorial gauge theory with gauge group G = SU(3) and N_f flavors of fermions ψ_i, $i = 1, ..., N_f$ transforming in the fundamental (triplet) representation. We impose the condition of asymptotic freedom (AF) for the self-consistency of the perturbative calculation of β. For simplicity, we take the fermions to be massless [2]. This theory is quantum chromodynamics (QCD) with N_f massless quarks.

The beta function of this theory has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell\, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell\, \alpha^\ell \ , \qquad (1)$$

where $a = g^2/(16\pi^2) = \alpha/(4\pi)$, b_ℓ is the ℓ-loop coefficient, $\bar{b}_\ell = b_\ell/(4\pi)^\ell$, and we extract a minus sign for convenience. The n-loop ($n\ell$) beta function, denoted $\beta_{n\ell}$, is obtained from Eq. (1) by changing the upper limit on the ℓ-loop summation from ∞ to n. The (scheme-independent) one-loop and two-loop coefficients are $b_1 = 11-(2/3)N_f$ [3] and $b_2 = 102-(38/3)N_f$ [4]. The AF condition implies the upper bound $N_f < N_{f,b1z} = 33/2$ [5], i.e., the integer upper bound $N_f \le 16$, which we impose. We denote the interval $0 \le N_f \le 16$ as I_{AF}. The b_ℓ with $\ell \ge 3$ are scheme-dependent [6]; b_3 and b_4 were calculated in [7] and [8] (and checked in [9]), in the $\overline{\rm MS}$ scheme [10]; e.g., $b_3 = (2857/2)-(5033/18)N_f + (325/54)N_f^2$. As $N_f \in I_{AF}$ increases from 0, b_2 decreases, vanishing at $N_{f,b2z} = 153/19 = 8.05$, and is negative in the real interval $153/19 < N_f < 33/2$, i.e., the integer interval I_{IRZ} : $9 \le N_f \le 16$. If $N_f \in I_{IRZ}$, then the two-loop beta function $\beta_{2\ell}$ has an IR zero (IRZ), at $\alpha = \alpha_{IR,2\ell} = -4\pi b_1/b_2$. Here we denote the IR zero (if it exists) of the n-loop beta function $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$. For N_f near the upper end of I_{IRZ}, $\alpha_{IR,2\ell}$ is small and can be studied perturbatively [4, 11]. As $N_f \in I_{IRZ}$ decreases, $\alpha_{IR,2\ell}$ increases toward strong coupling. Hence, to study the IR zero for N_f toward the middle and lower part of I_{IRZ} with reasonable accuracy, one requires higher-loop calculations. These were carried out to four-loop order in [12]-[16] [17]. Clearly, such a perturbative calculation of the IR zero of $\beta_{n\ell}$ is only reliable if the resultant $\alpha_{IR,n\ell}$ is not excessively large. Since the b_ℓ with $\ell \ge 3$ are scheme-dependent, it is necessary to assess the sensitivity of the value obtained for $\alpha_{IR,n\ell}$ for $n \ge 3$ to the scheme used for the calculation. This was done in [18]-[21] (see also [22, 23]). In [18]-[19], a set of conditions that an acceptable scheme transformation must satisfy were presented, and it was shown that although these are automatically satisfied in the local vicinity of the origin, $\alpha = 0$ (as in optimized schemes for perturbative QCD calculations [24, 25]), they are not automatically satisfied, and indeed, are quite restrictive conditions, when one applies the scheme transformation at an IR zero away from the origin.

Here we report the first calculation of the five-loop IR zero of β and resultant five-loop evaluation of the anomalous dimension of the fermion bilinear at this IR zero, for $N_f \in I_{IRZ}$, making use of the recent calculation of b_5 in the $\overline{\rm MS}$ scheme from [26]. The results are of fundamental importance for understanding the RG evolution of SU(3) gauge theory with variable fermion content.

The anomalous dimension γ_m of the fermion bilinear operator $\bar{\psi}_i\psi_i$ (no sum on i) is defined as $D(\bar{\psi}_i\psi_i) = 3-\gamma_m$, where D is the full scaling dimension. Knowing $\alpha_{IR,n\ell}$, one can then evaluate γ_m (calculated to the same n-loop order) at $\alpha = \alpha_{IR,n\ell}$; we denote this as $\gamma_{IR,n\ell}$. This anomalous dimension is of particular interest, since (if calculated to all orders) it is a scheme-independent physical quantity. (Unless indicated otherwise hereafter, the scheme taken for the b_n and resultant $\beta_{n\ell}$, $\alpha_{IR,n\ell}$, and $\gamma_{IR,n\ell}$ with $n \ge 3$ is the $\overline{\rm MS}$ scheme.)

Our previous work showed the usefulness of higher-loop calculations of $\gamma_{IR,n\ell}$. For example, for a (vectorial) SU(3) gauge theory with $N_f = 12$ massless Dirac fermions, the values of $\gamma_{IR,n\ell}$ at the two-loop, three-loop, and four-loop level were found to be 0.773, 0.312, and 0.253, respectively [14, 15]. Our four-loop result, $\gamma_{IR,4\ell}$, is in good agreement with the fully nonperturbative lattice calculations $\gamma_{IR} = 0.27 \pm 0.03$ [27], $\gamma_{IR} \simeq 0.25$ [28], and $\gamma_{IR} = 0.235 \pm 0.046$ [29],[30]. These measurements are part of an intensive lattice program to elucidate the properties of asymptotically free gauge theories with various fermion contents, in particular, those exhibiting quasiconformal behavior; besides their intrinsic field-theoretic interest, such theories might play a role in physics beyond the Standard Model [30]. Similar agreement was found for four-loop calculations in other schemes [18]-[21]. An iterative method to calculate γ_{IR} in a scheme-independent manner has been presented in [22]. It allows for a direct comparison of perturbative methods with exact results in $\mathcal{N}=1$ supersymmetric QCD, for which it was shown that γ_{IR} is very well described already at a few loops level throughout the entire conformal interval.

In the UV to IR evolution, as μ decreases, $\alpha(\mu)$ approaches the IR zero in β. If this zero occurs at relatively weak coupling, it can be an exact IR fixed point (IRFP) of the RG, and the corresponding IR phase is a chirally symmetric, deconfined non-Abelian Coulomb phase (NACP, conformal interval). If the IR zero in β occurs at a sufficiently large value of α, then the IR phase has confinement and spontaneous chiral symmetry breaking (SχSB) associated with a nonzero bilinear fermion condensate formed at a scale Λ. In this case, the fermions gain dynamical masses and are integrated out of the low-energy effective theory applicable for $\mu < \Lambda$. The IR zero in β is then only an approximate IRFP and similarly, γ_{IR} is only an effective quantity describing the RG flow near this approximate IRFP.

We next describe the behavior of b_5 as a function of N_f (for the behavior of b_3 and b_4, see [14, 16].) As $N_f \in I_{AF}$ increases from 0, b_5 initially decreases through positive values, reaches a minimum at $N_f = 6.074$ [5], where $\bar{b}_5 = 0.640 \times 10^{-3}$, and then increases. For all $N_f \in I_{AF}$, b_3 is negative-definite, while b_4 and b_5 are positive-definite. We list values of the $\bar{b}_\ell$ for $1 \le \ell \le 5$ in Table I.

TABLE I: Values of the $\bar{b}_\ell$ for $1 \le \ell \le 5$ as a function of N_f, with b_ℓ for $\ell = 3, 4, 5$ calculated in the $\overline{\text{MS}}$ scheme.

N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$
0	0.875	0.646	0.720	1.173	1.714
1	0.822	0.566	0.582	0.910	1.175
2	0.769	0.485	0.450	0.681	0.744
3	0.716	0.405	0.324	0.485	0.416
4	0.663	0.325	0.205	0.322	0.186
5	0.610	0.245	0.091	0.194	0.0494
6	0.557	0.165	−0.016	0.099	0.000866
7	0.504	0.084	−0.118	0.039	0.0354
8	0.451	0.004	−0.213	0.015	0.1475
9	0.398	−0.076	−0.303	0.025	0.332
10	0.345	−0.156	−0.386	0.072	0.583
11	0.292	−0.236	−0.463	0.154	0.894
12	0.239	−0.317	−0.534	0.273	1.261
13	0.186	−0.397	−0.599	0.429	1.676
14	0.133	−0.477	−0.658	0.622	2.134
15	0.080	−0.557	−0.711	0.852	2.628
16	0.0265	−0.637	−0.758	1.121	3.152

For our analysis of the IR zero of β, it is convenient to extract a prefactor and define a reduced n-loop beta function as

$$\beta_{r,n\ell} \equiv \frac{\beta_{n\ell}}{-2\alpha^2 \bar{b}_1} = 1 + \sum_{\ell=2}^{n} \bar{\rho}_\ell \, \alpha^{\ell-1} \tag{2}$$

where $\bar{\rho}_\ell = \bar{b}_\ell / \bar{b}_1$. The equation $\beta_{r,n\ell} = 0$ determines the IR zero and is a polynomial equation of degree $n-1$ in α. Among the $n-1$ roots, the smallest positive (real) root, if there is such a root, is $\alpha_{IR,n\ell}$. The nature of the roots at the $n=3$ and $n=4$ loop level has been discussed in [14, 15].

We present our results for $\alpha_{IR,5\ell}$ in Table II. We begin the discussion at the upper end of the interval I_{IRZ}. For $14 \le N_f \le 16$, we find that $\alpha_{IR,5\ell}$ is close to, and slightly larger than, $\alpha_{IR,4\ell}$. For $N_f = 13$, $\alpha_{IR,5\ell}$ is about 20 % larger than $\alpha_{IR,4\ell}$. If $9 \le N_f \le 12$, we find that the five-loop beta function (in the $\overline{\text{MS}}$ scheme, with b_5 from [26]) has no physical IR zero; instead, the roots of the quartic polynomial $\beta_{r,5\ell}$ consist of two complex-conjugate (c.c.) pairs. This is a surprising result, since at all of the lower-loop orders, namely $n=2$, $n=3$, and $n=4$, for $N_f \in I_{IRZ}$, the n-loop beta functions (in this $\overline{\text{MS}}$ scheme and also other schemes [18]-[21]) have physical IR zeros $\alpha_{IR,n\ell}$, and one would naturally expect that as one extends the calculation of $\beta_{n\ell}$ to higher-loop order, this behavior would continue. Specifically, we find the following: $N_f = 9 \Rightarrow \alpha_{IR,5\ell} = 0.863 \pm 0.515i$; $N_f = 10 \Rightarrow \alpha_{IR,5\ell} = 0.715 \pm 0.382i$; $N_f = 11 \Rightarrow \alpha_{IR,5\ell} = 0.609 \pm 0.277i$; and $N_f = 12 \Rightarrow \alpha_{IR,5\ell} = 0.528 \pm 0.176i$. Although these roots are unphysical if $9 \le N_f \le 12$, the respective real parts are similar to lower-loop values; for example, $\text{Re}(\alpha_{IR,5\ell}) = 0.609$ for $N_f = 11$, which is close to $\alpha_{IR,4\ell} = 0.626$, etc. As N_f increases in this interval $9 \le N_f \le 12$, the real part and the magnitude of the imaginary part decrease, consistent with the approach to the real value $\alpha_{IR,5\ell} = 0.406$ at $N_f = 13$. Formally extending N_f to real numbers, we find that as N_f approaches the value $N_f \simeq 12.8944$ from below, the two complex-conjugate roots approach the real axis, with the real part approaching 0.47, and for larger $N_f \in I_{IRZ}$, the two c.c. roots are replaced by two real roots, which respectively decrease and increase from $\alpha_{IR,5\ell} \simeq 0.47$ as N_f increases beyond 12.8944. At the next physical integer value, $N_f = 13$, the lower root in this pair occurs at $\alpha_{IR,5\ell} = 0.406$, as listed in Table II, while the upper one occurs at 0.5195.

A necessary condition for the perturbative calculation of the IR zero to be reliable is that the magnitude of the fractional difference

$$\Delta_{IR;n-1,n} = \frac{\alpha_{IR,(n-1)\ell} - \alpha_{IR,n\ell}}{\frac{1}{2}[\alpha_{IR,(n-1)\ell} + \alpha_{IR,n\ell}]} \tag{3}$$

TABLE II: Values of $\alpha_{IR,n\ell}$ as a function of N_f for $N_f \in I_{IRZ}$ and loop order $2 \le n \le 5$. See text for discussion of $\alpha_{IR,5\ell}$ for $9 \le N_f \le 12$.

N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$	$\alpha_{IR,5\ell}$
9	5.24	1.028	1.072	−
10	2.21	0.764	0.815	−
11	1.23	0.578	0.626	−
12	0.754	0.435	0.470	−
13	0.468	0.317	0.337	0.406
14	0.278	0.215	0.224	0.233
15	0.143	0.123	0.126	0.127
16	0.0416	0.0397	0.0398	0.0398

TABLE III: Values of $\Delta_{IR;n-1,n}$ as a function of N_f for $N_f \in I_{IRZ}$. See text for discussion of $\Delta_{IR;4,5}$ for $9 \le N_f \le 12$.

N_f	$\Delta_{IR;2,3}$	$\Delta_{IR;3,4}$	$\Delta_{IR;4,5}$
9	1.344	−0.04175	−
10	0.971	−0.0642	−
11	0.723	−0.0791	−
12	0.537	−0.0785	−
13	0.386	−0.0639	−0.185
14	0.258	−0.0415	−0.0404
15	0.146	−0.0185	−0.00770
16	0.0461	−0.00255	−0.000288

should be reasonably small and should tend to decrease with increasing loop order, n [31]. We have calculated the various $\Delta_{IR,n-1,n}$ and list the values in Table III. As is evident, this necessary condition is satisfied if $14 \le N_f \le 16$. If $N_f = 13$, then the requisite behavior is observed for $\Delta_{IR;23}$ and $|\Delta_{IR;34}|$, but $|\Delta_{IR;45}|$ is actually about three times larger than $|\Delta_{IR;34}|$. For lower values of $N_f \in I_{IRZ}$, the $|\Delta_{IR;n-1,n}|$ criterion is not applicable, since $\beta_{5\ell}$ is complex.

These results are a consequence of the properties of the relevant coefficients $\bar{b}_n$ in β. In general, if, as a function of $N_f \in I_{IRZ}$, $|\bar{b}_n|$ becomes very small in magnitude, then the n-loop contribution to β will tend to be a commensurately small correction to the $(n-1)$-loop beta function, so $\Delta_{IR;n-1,n}$ will also be small. As N_f decreases from 16 to 9, $|\bar{b}_3|$ decreases by a factor of 2.5 and $\bar{b}_4$ decreases sharply, by a factor of 45. This strong decrease in $\bar{b}_4$ means that although the overall size of $\alpha_{IR,4\ell}$ increases as N_f decreases in this interval I_{IRZ}, the fractional difference $\Delta_{IR;3,4}$ remains small, as is evident in Table III. In contrast, although $\bar{b}_5$ also decreases as N_f decreases in I_{IRZ}, it is still considerably larger than $\bar{b}_4$, leading to the larger value of $|\Delta_{IR;4,5}|$ observed for $N_f = 13$.

Our calculation of $\alpha_{IR,5\ell}$ thus reveals new complexities with the IR zero in β for $N_f \in I_{IRZ}$ that were not observed at lower-loop level and hence were not anticipated at five-loop order, since one expects that (in a nonpathological scheme) calculations at higher-loop order should exhibit greater stability than those at lower-loop order [31]. In view of our finding, we next make use of the powerful method of Padé approximants (PAs) [32] to study the IR zero in β at the five-loop level. The $[p, q]$ PA to $\beta_{r,n\ell}$ is the rational function

$$[p, q]_{\beta_{r,n\ell}} = \frac{1 + \sum_{j=1}^{p} n_j \alpha^j}{1 + \sum_{k=1}^{q} d_k\, \alpha^k} \tag{4}$$

with $p+q = n-1$, where the n_j and d_j are α-independent coefficients. For a given $\beta_{r,n\ell}$, there are thus n PAs, namely the set $\{\ [n-k, k-1]_{\beta_{r,n\ell}}\ \}$ with $1 \le k \le n$. For $n = 5$ loops, this is the set $\{[4, 0], [3, 1], [2, 2], [1, 3], [0, 4]\}$. The [4,0] PA is just $\beta_{r,5\ell}$ itself, which we have already analyzed, and the [0,4] PA has no zero and hence cannot be used for the analysis of the IR zero of $\beta_{r,5\ell}$, which leaves us with the remaining three PAs. We have calculated and analyzed these. If a $[p, q]_{\beta_{r,n\ell}}$ PA has a physical IR zero at this $n = 5$ loop level, it is denoted as $\alpha_{IR,5\ell,[p,q]}$. Clearly, if a PA has a pole closer to the origin (indicated as *pcl*) than a zero, then this zero is not a reliable guide to the UV to IR evolution of the theory from weak coupling. Furthermore, a PA may contain an essentially coincident pair of a zero and pole (indicated by *zp*); in this case, the zero and pole factors cancel and may be neglected.

We present the results of our Padé analysis in Table IV. Importantly, we find that in several cases the PAs yield results for the IR zero at the five-loop level that are physical and/or more stable than the zeros of $\beta_{r,5\ell}$ themselves. For $N_f = 16$ and $N_f = 15$, all of the three $\alpha_{IR,5\ell,[p,q]}$ listed in Table IV agree very well with the respective values of $\alpha_{IR,5\ell}$, and this is also true for $\alpha_{IR,5\ell,[2,2]}$ and $\alpha_{IR,5\ell,[1,3]}$ in the case of $N_f = 14$. For $N_f = 13$, the values of $\alpha_{IR,5\ell,[2,2]}$ and $\alpha_{IR,5\ell,[1,3]}$ lie roughly midway between $\alpha_{IR,4\ell}$ and $\alpha_{IR,5\ell}$. For $9 \le N_f \le 12$, where there is no physical IR zero of $\beta_{r,5\ell}$, at least one of the PAs, namely $[3, 1]_{\beta_{r,5\ell}}$, yields physical IR zeros, and the respective values of $\alpha_{IR,5\ell,[3,1]}$ are reasonably close to, and somewhat smaller than, the corresponding values of $\alpha_{IR,4\ell}$. (PAs that yield negative or complex zeros are marked with −.) Thus, using the physical results from the Padé approximants helps to circumvent the problem with complex $\alpha_{IR,5\ell}$ in this lower region of I_{IRZ}.

The anomalous dimension γ_m has the series expansion $\gamma_m = \sum_{\ell=1}^{\infty} c_\ell a^\ell$. The n-loop γ_m is $\gamma_{m,n\ell} = \sum_{\ell=1}^{n} c_\ell a^\ell$. The coefficient $c_1 = 8$ is scheme-independent, while the c_ℓ with $\ell \ge 2$ are scheme-dependent [6]. In the $\overline{\rm MS}$ scheme, the c_ℓ have been calculated up to $\ell = 4$ [33] and recently to $\ell = 5$ [34]; e.g., $c_2 = (404/3) - (40/9)N_f$, etc.

As noted above, we define $\gamma_{IR,n\ell} = \gamma_{n\ell}$ evaluated at $\alpha = \alpha_{IR,n\ell}$. We calculate $\gamma_{IR,5\ell}$ here. For $14 \le N_f \le 16$, we use our values of $\alpha_{IR,5\ell}$. For $N_f = 13$, we use $\alpha = \alpha_{IR,5\ell,[1,3]}$ and for $10 \le N_f \le 12$ we use $\alpha = \alpha_{IR,5\ell,[3,1]}$. In both the chirally symmetric and chirally broken IR phases, the IR value of γ_m has the upper bound [35] $\gamma_{IR,n\ell} < 2$. Since $\gamma_{IR,2\ell}$ violates this for $N_f = 10$ [14], we

TABLE IV: Values of $\alpha_{IR,n\ell,[p,q]}$ from $[p,q]$ Padé approximants to $\beta_{r,5\ell}$, as a function of $N_f \in I_{IRZ}$, including comparison with $\alpha_{IR,4\ell}$ and $\alpha_{IR,5\ell}$. The symbols (i) *zp* and (ii) *pcl* mean that the Padé approximant has (i) a coincident zero-pole pair closer to the origin, (ii) a pole or complex-conjugate pair of poles closer to the origin in the complex α plane. Entries with − are unphysical.

N_f	$\alpha_{IR,4\ell}$	$\alpha_{IR,5\ell}$	$\alpha_{IR,5\ell,[3,1]}$	$\alpha_{IR,5\ell,[2,2]}$	$\alpha_{IR,5\ell,[1,3]}$
9	1.072	−	1.02_{zp}	−	−
10	0.815	−	0.756_{zp}	−	*pcl*
11	0.626	−	0.563_{zp}	−	*pcl*
12	0.470	−	0.4075_{zp}	0.634	0.614
13	0.337	0.406	−	0.376	0.375
14	0.224	0.233	−	0.232	0.232
15	0.126	0.12 7	0.127	0.127	0.127
16	0.0398	0.0398	0.0398	0.0398	0.0398

TABLE V: Values of the five-loop anomalous dimension for the fermion bilinear, $\gamma_{IR,5\ell}$, evaluated at the IR zero of the five-loop beta function, $\beta_{5\ell}$, as a function of N_f for $11 \le N_f \le 16$, including comparison with lower-loop values of $\gamma_{IR,n\ell}$.

N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$	$\gamma_{IR,5\ell}$
11	1.61	0.439	0.250	0.294
12	0.773	0.312	0.253	0.255
13	0.404	0.220	0.210	0.239
14	0.212	0.146	0.147	0.154
15	0.0997	0.0826	0.0836	0.0843
16	0.0272	0.0258	0.0259	0.0259

only show results for $11 \le N_f \le 16$. These are given in Table V. For N_f values where the five-loop IR zero occurs at sufficiently weak coupling, our new five-loop value for the anomalous dimension at this zero is close to the four-loop value. In particular, our value $\gamma_{IR,5\ell} = 0.255$ at $N_f = 12$ is in good agreement with lattice measurements of this quantity, as was our value $\gamma_{IR,4\ell} = 0.253$ in [14].

In summary, using the recently calculated five-loop term in the SU(3) beta function from [26], we have presented the first calculation of the five-loop IR zero in the beta function for an SU(3) gauge theory and the first five-loop calculation of the anomalous dimension of the fermion bilinear operator at this IR zero.

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